International Journal of Theoretical Physics, Vol. 31, No. 12, 1992

Solving Bertolami's Equations for Brans–Dicke Cosmological Models

Zoran Pazameta[1]

Received August 14, 1991

Solutions are developed for Berman and Som's formulation of Bertolami's equations for a Brans–Dicke cosmology with time-dependent cosmological term. Physical constraints are applied to these solutions to deduce conditions necessary for constructing plausible cosmological models in this theory.

An interesting set of equations was formulated by Bertolami (1986) for a Brans–Dicke cosmology with a Robertson–Walker metric, a perfect-fluid source, and a time-dependent cosmological term. Berman and Som (1990) subsequently obtained conditions for solving these equations for the general equation of state

$$p=\alpha\rho \tag{1}$$

where α is a constant. Bertolami had solved his equations for the particular cases $\alpha=0$ and $\alpha=1/3$ (i.e., $p=0, p=\rho/3$), and had obtained, among other results, that

$$\Lambda=Et^{-2} \tag{2}$$

where t is cosmic time and E is a constant. Relation (2) was adopted by Berman and Som, who also employed the expression

$$R(t)=(mDt)^{1/m} \tag{3}$$

for the scale factor $R(t)$, where m and D are nonzero constants and m is related to the (constant) deceleration parameter q by

$$m=q+1 \tag{4}$$

[1]Department of Physical Sciences, Eastern Connecticut State University, Willimantic, Connecticut 06226.

0020-7748/92/1200-2115$06.50/0

Inserting (1) and (3) into the appropriate equation of Bertolami, they obtained

$$\rho = Ct^{-3(1+\alpha)/m} \tag{5}$$

for the density. Selecting

$$\phi = St^{A} \tag{6}$$

for the Brans–Dicke scalar field (C, S, and A being constants), they set

$$A = 2 - 3(1+\alpha)/m \tag{7}$$

in order to obtain, from the remaining equations of Bertolami, three relations for determining all of the above constants, as well as the Brans–Dicke constant w, in terms of α and m. We write these relations as

$$1 + \frac{k(mt)^{2-2/m}}{D^{2/m}} = \frac{8\pi C}{3S}\left(\frac{4+2w}{3+2w}\right)m^2 + \frac{wA^2}{6}m^2 - Am + Em^2 \tag{8}$$

$$A\left(A + \frac{3}{m} - 1\right) + \frac{2E}{3+2w}\left(1 + \frac{2}{A}\right) = \frac{8\pi C}{S}\,\frac{4+2w}{(3+2w)^2}(1-3\alpha) \tag{9}$$

$$\frac{8\pi C}{S}\left(\frac{4+2w}{3+2w}\right)(3-A) + wA^2\left(\frac{3}{2m} - 1\right) + \frac{3A}{m}\left(2 - \frac{3}{m}\right) = E\left(\frac{3}{m} - 2\right) \tag{10}$$

Our aim is to solve the above system of equations for the three cases $k = 0, \pm 1$, in order to obtain w in terms of α and m; the constants A, E, and $8\pi C/S$ then follow from w and equations (8)–(10).

For $k = 0$, the time-dependent term in (8) vanishes, and the relations (8)–(10) give

$$w = P_1(m, \alpha)/P_2(m, \alpha) \tag{11}$$

where

$$P_1(m, \alpha) = \sum_{i=0}^{4} a_i(m)\alpha^i \tag{12}$$

with

$$\begin{aligned}
a_0 &= 117 - 384m + 375m^2 - 116m^3 - 4m^4 \\
a_1 &= 216 - 390m + 78m^2 + 84m^3 \\
a_2 &= 288 - 144m - 117m^2 \\
a_3 &= 432 - 162m \\
a_4 &= 243
\end{aligned} \tag{13}$$

and where

$$P_2(m, \alpha) = \sum_{i=0}^{4} b_i(m)\alpha^i \tag{14}$$

with

$$\begin{aligned}
b_0 &= -54 + 243m - 306m^2 + 148m^3 - 24m^4 \\
b_1 &= -189 + 369m - 96m^2 - 92m^3 + 32m^4 \\
b_2 &= -189 - 171m + 450m^2 - 144m^3 \\
b_3 &= -27 - 405m + 216m^2 \\
b_4 &= 27 - 108m
\end{aligned} \tag{15}$$

Thus far, m and α are arbitrary. We can, however, estimate a range of values for each, based on our present-day knowledge. A value often used for the deceleration parameter is (Stephani, 1985)

$$q = 1 \pm 1 \tag{16}$$

which, with (4), gives a range for m of

$$1 \leq m \leq 3 \tag{17}$$

(We note that, to recover the results of Bertolami (1986) for $\alpha = 0$ and $\alpha = 1/3$, we would set $m = 1$.) In addition, cosmological matter obeying relation (1) has values of α (Stephani, 1985) which are restricted to the range

$$0 \leq \alpha \leq 1/3 \tag{18}$$

There also exist restrictions on the Brans–Dicke constant w. If we require that the theory recover Einstein's result for the gravitational bending of light, then, as is well known,

$$\frac{4 + 2w}{3 + 2w} > 0 \tag{19}$$

If, in addition, the theory is required to give Einstein's result for the perihelion precession of Mercury, the stronger restriction

$$w \geq 6 \tag{20}$$

applies (Lord, 1979). When used with equations (11)–(13), this restriction leads to the condition

$$\sum_{i=0}^{4} (a_i - 6b_i)\alpha^i \geq 0 \tag{21}$$

which must be satisfied for any choice of m and α if the solution for $k=0$ is to produce a plausible cosmological model. Such a model is easily constructed, once m and α are selected, by using (11) and any two of equations (8)–(10) to find the remaining constants of the theory.

Turning now to the models with $k=\pm 1$, we see that equation (8) loses its time dependence only if we set

$$m=1 \tag{22}$$

Then,

$$A=-(1+\alpha) \tag{23}$$

and equations (8)–(10) are solved to give

$$w=-[Q_1(\alpha)+Q_2(\alpha)]/Q_3(\alpha) \tag{24}$$

where

$$\begin{aligned} Q_1(\alpha)&=3(27\alpha^3+39\alpha^2+16\alpha+4)\\ Q_2(\alpha)&=9(1+\alpha)k/D^2\\ Q_3(\alpha)&=27\alpha^3+81\alpha^2+45\alpha+7 \end{aligned} \tag{25}$$

If we impose condition (20) on w, equations (24) and (25) give

$$D^2\geq 3k(1+\alpha)/(27\alpha^3+123\alpha^2+74\alpha+10) \tag{26}$$

For $k=+1$, this allows us to estimate a minimum value for D^2, given a particular value of $\alpha\geq 0$. However, for $k=-1$, the condition (26) is satisfied for all α, giving a cosmological model valid for all D^2.

Finally, we note that equations (9) and (10) give a particularly simple expression for the coefficient of the cosmological term:

$$E=-(1+3\alpha)^2\left[\frac{(5+3\alpha)}{(1+\alpha)}\frac{w}{6}+1\right] \tag{27}$$

so that $E<0$ for all α if $k=-1$, and also for $k=+1$ if condition (26) is satisfied (or, indeed, whenever $w\geq 0$).

In conclusion, we have found that, within the theoretical framework of Berman and Som, Bertolami's equations for a Brans–Dicke cosmology with time-dependent cosmological term can be solved to give values for the constants of the theory in terms of m and α (for $k=0$) or just α (for $k=\pm 1$). By applying physically reasonable constraints to the values of α and the Brans–Dicke constant w, we found that plausible cosmological models may result from the $k=0$ solution only if m and α satisfy a particular condition—from the $k=+1$ solution only if the parameter D^2 is greater than a certain value, and from the $k = -1$ solution for all allowed values of α and all values of D^2.

REFERENCES

Berman, M. S., and Som, M. M. (1990). *International Journal of Theoretical Physics*, **29**, 1411.
Bertolami, O. (1986). *Nuovo Cimento*, **93B**(1), 36.
Lord, E. A. (1979). *Tensors, Relativity and Cosmology*, Tata McGraw-Hill, New Delhi, p. 193.
Stephani, H. (1985). *General Relativity*, Cambridge University Press, Cambridge, p. 260.